EQUILIBRIUM AND STABILITY IN A HIGH-CURRENT DISCHARGE IN A DENSE PLASMA UNDER CONDITIONS OF RADIATIVE CONDUCTION

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Equilibrium and stability are examined for a high-current self-compressed discharge in a dense, optically opaque plasma of finite conductivity, with allowance for dissipation via radiative heat transfer. If the thermal conductivity is high, the plasma temperature is virtually constant throughout the cross-section of the discharge, whereas the density and pressure fall off fairly rapidly away from the axis. The spectrum for small oscillations shows that such an equilibrium discharge is unstable with respect to short-wave hydrodynamic oscillations (bending and necking) if the plasma conductivity is low. Instability can develop only for long-wave perturbations in a cylindrical discharge, and also for a nonequilibrium discharge when the rise time is less than the equilibration time. A planar equilibrium discharge is stable, while a cylindrical equilibrium discharge in a dense low-temperature plasma is more stable than one in a high-temperature plasma.

There have been several discussions of the use of high-current discharges in dense plasmas as light sources for laser pumping. The choice of discharge dimensions is governed by the temperature T of the radiating surface, which should be 3-10 eV. Only ohmic heating can allow one to keep a plasma at such a temperature for a sufficiently long time (around 100 μsec). On the other hand, hydrodynamic instabilities (bends, necks, hot spots) can arise in a dense plasma carrying a current, which can lead to current interruption and plasma dispersal (see [1] for literature). Stability is therfore a major problem in the use of such discharges as light sources. However, it is not correct to apply the theory of [1] to such discharges, since this theory is for a not very dense, hot, transparent plasma under conditions such that radiation does not play a major part in the development of the discharge, whereas a discharge in a dense, optically opaque plasma is best as a light source. Such a plasma can have considerable radiative energy transfer, which can influence the entire character of the discharge. Moreover, effects due to the finite conductivity (diffusion of electric and magnetic fields) may play major parts at these relatively low temperatures. Here we present a theoretical discussion of the equilibrium and stability of a high-current discharge in a dense, optically opaque plasma having a finite conductivity and considerable radiative heat transfer.

1. Formulation. The following is [2] the complete system of equations of magnetohydrodynamics for a plasma with allowance for radiative heat transfer:

$$\operatorname{div}\mathbf{B}=0,\qquad \operatorname{rot}\mathbf{B}=\frac{4\pi}{c}\mathbf{j}=\frac{4\pi}{c}\sigma\left\{\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right\},$$

$$-c\operatorname{rot}\mathbf{E}=\frac{\partial\mathbf{B}}{\partial t}=\operatorname{rot}(\mathbf{v}\times\mathbf{B})-\frac{c^2}{4\pi}\operatorname{rot}\left(\frac{1}{\sigma}\operatorname{rot}\mathbf{B}\right),$$

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}+\rho\varepsilon+\frac{B^2}{8\pi}\right)+\operatorname{div}(\mathbf{q}+\mathrm{S})=0,$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=$$

$$=-\nabla P+\eta\Delta\mathbf{v}+\left(\xi+\frac{\eta}{3}\right)\operatorname{grad}\operatorname{div}\mathbf{v}+\frac{1}{4\pi}\operatorname{rot}\mathbf{B}\times\mathbf{B},$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,\qquad P=P(\rho,\ T). \tag{1.1}$$

Here q is the energy flux and S is the radiation flux in an optically opaque plasma [3]:

$$\mathbf{q}=\rho\mathbf{v}\left(\frac{v^2}{2}+\varepsilon+\frac{P}{\rho}\right)+\frac{1}{4\pi}\mathbf{B}\times(\mathbf{v}\times\mathbf{B})+$$

$$+\frac{c^2}{(4\pi)^2\sigma}\operatorname{rot}\mathbf{B}\times\mathbf{B}-(\mathbf{v}\cdot\sigma')-\chi\nabla T$$

$$\mathrm{S}=-{}^{16}/_{3}\sigma^\circ T^3 l(\rho,\ T)\nabla T \tag{1.2}$$

$\sigma^\circ = 5.67\cdot10^{-5}$ erg/cm^2-deg^4-sec is Stefan's constant and l is the Rosseland path length for the light.

If T is 3–10 eV, the plasma may be considered as a completely ionized ideal gas, so

$$P=(1+z)N\varkappa T=\frac{(1+z)\varkappa}{M}\rho T,$$

$$\varepsilon=c_v T=\frac{3}{2}\frac{(1+z)\varkappa}{M}T=\frac{3}{2}\frac{P}{\rho}, \tag{1.3}$$

in which M is the ion mass and z is effective ion charge. The conductivity is then $\sigma=\alpha z^{-1}T^{3/2}$, with $\alpha=4\cdot10^7$.

In (1.1) we have neglected the radiation energy relative to the internal (thermal) energy, which is correct if

$$\frac{\sigma^\circ T^4}{c}\ll P\approx\rho\varepsilon. \tag{1.4}$$

This condition is obeyed closely for $N\geq10^{16}$ cm^{-3} in the above temperature range. We subsequently neglect the electron thermal conductivity relative to the radiative one, i.e., we assume that

$$\chi\ll\frac{16}{3}\sigma^\circ T^3 l. \tag{1.5}$$

This inequality also allows us to neglect the viscous terms in (1.1) and (1.2).

All the results below on the stability are independent of the explicit form of l (ρ, T), but to estimate the plasma parameters we use the expression [3]

$$l=\gamma_0\frac{M^2T^{7/2}}{z(1+z)^2\rho^2}, \tag{1.6}$$

which is correct for a gas with highly ionized atoms* for $\gamma_0\approx4.4\cdot10^{22}$.

We use (1.6) with $\chi\approx e^{-2}\varkappa^2\sigma T$ to write (1.5) as

$$N\ll10^{12}\,T^2. \tag{1.7}$$

This inequality is obeyed closely for $N<10^{21}$ cm^{-3} in this range in T.

*Expression (1.6) is also applicable to description of free-free transitions in a completely ionized plasma for $\gamma_0\approx4.8\cdot10^{24}$.

2. Discharge equilibrium in a dense plasma. System (1.1) shows that the electric field E_0 should be considered uniform over the cross-section of the plasma (for $v_0 = 0$) in the strictly equilibrium steady state. The pressure, density, temperature, current, and magnetic field are, in general, functions of the coordinates.

The following equations* (with subscript 0 denoting equilibrium quantities) define the spatial distribution of these quantities:

$$\operatorname{rot}\mathbf{B}_0 = \frac{4\pi}{c}\sigma_0\mathbf{E}_0 = \frac{4\pi\alpha}{cz}T_0^{3/2}\mathbf{E}_0,$$

$$\nabla P_0 = \frac{1}{4\pi}\operatorname{rot}\mathbf{B}_0\times\mathbf{B}_0,$$

$$\operatorname{div}\left\{\frac{c^2}{(4\pi)^2\sigma_0}\operatorname{rot}\mathbf{B}_0\times\mathbf{B}_0 - \frac{16}{3}\sigma^\circ T_0^3 l_0\nabla T_0\right\} = 0,$$

$$P_0 = \frac{(1+z)\varkappa}{M}\rho_0 T_0. \tag{2.1}$$

Here we consider discharges of two types: a planar (surface) discharge and a simple cylindrical one (Z pinch). Only numerical methods can give exact solutions to (2.1) in both cases, but we do not need an exact solution for our purposes, because a discharge in a dense plasma can be used efficiently as a light source only when the plasma surface has a high T, which will occur when T may be taken as almost constant over the cross-section. We assume that this is so to get from the first three equations in (2.1) for the two cases, respectively, that

$$B_0 = \sqrt{8\pi P_0(0)}\,\frac{x}{x_p},\quad P_0 = P_0(0)\left(1-\frac{x^2}{x_p^2}\right),$$

$$x_p^2 = \frac{P_0(0)\,c^2}{2\pi\sigma_0^2(0)\,E_0^2},$$

$$B_0 = \sqrt{4\pi P_0(0)}\,\frac{r}{r_p},\quad P_0 = P_0(0)\left(1-\frac{r^2}{r_p^2}\right),$$

$$r_p^2 = \frac{P_0(0)\,c^2}{\pi\sigma_0^2(0)\,E_0^2}. \tag{2.2}$$

Here $P_0(0)$ and $\sigma_0(0)$ are the equilibrium P and σ (conductivity) at the axis. We use (2.2) and (1.6) to get T_0 from the last equation in (2.1) as

$$T_0 = T_0(0)\left\{1-\frac{x^2}{x_T^2}\left(1-\frac{x^2}{x_p^2}+\frac{x^4}{3x_p^4}\right)\right\},$$

$$T_0 = T_0(0)\left\{1-\frac{r^2}{r_T^2}\left(1-\frac{r^2}{r_p^2}+\frac{r^4}{3r_p^4}\right)\right\}, \tag{2.3}$$

in which

$$\frac{x_T^2}{x_p^2} = \frac{r_T^2}{r_p^2} = \frac{16}{3}\sigma^\circ\frac{4\pi\gamma_0\varkappa^2\sigma_0(0)\,T_0^{11/2}(0)}{zc^2 p_0^3(0)}. \tag{2.4}$$

We have from the above assumption that $x_T^2 \gg x_p^2$, to which we must add the condition for applicability of radiative heat transfer, $x_p \gg l$ (or $r_p \gg l$). These inequalities may be put as follows for z = 2:

$$10^7 T_0^{4/3} \gg N \gg 10^{10} T_0^{4/5} E^{2/5}. \tag{2.5}$$

Formulas (2.2) and (2.3) are correct for $x \le x_p$ (or $r \le r_p$). The pressure and density fall sharply near x_p, while the Rosseland length increases without bound, and the plasma becomes transparent. The region of transparency is negligibly small for $x_p \gg l$ and makes no substantial contribution to the energy balance of the discharge, which is governed on the one hand by ohmic heating and on the other by emission from the surface. Then $x_T^2 \gg x_p^2$, and the radiation flux from the surface is that from a black body at $T_0(0)$.

We have made substantial use of the uniformity of E_0 in examining the equilibrium state, but this assumption is correct only after the passage of an adequate time (the equilibration time) from application of the potential. Therefore, in examining the stability we are not bound to the particular equilibrium state of (2.2) and (2.3) and assume, in general, an arbitrary inhomogeneous current distribution in the quasi-stationary state.

3. Stability of a planar discharge. We consider the stability in the presence of small perturbations:

$$\rho\to\rho_0+\rho_1,\ T\to T_0+T_1,$$

$$P\to P_0+P_1,\ \mathbf{B}\to\mathbf{B}_0+\mathbf{B}_1,\ \mathbf{v}.$$

We linearize (1.1)–(1.3) by taking the perturbed quantities in the planar discharge as dependent on time and coordinates via $(f(x)\exp(-i\omega t + ik_y y + ik_z z)$, so

$$-i\omega\rho_1 + \operatorname{div}\rho_0\mathbf{v} = 0,\qquad \operatorname{div}\mathbf{B}_1 = 0,$$

$$-i\omega\rho_0\mathbf{v} = -\nabla P_1 + \frac{1}{4\pi}\operatorname{rot}\mathbf{B}_0\times\mathbf{B}_1 + \frac{1}{4\pi}\operatorname{rot}\mathbf{B}_1\times\mathbf{B}_0,$$

$$i\omega\left(\frac{3}{2}P_1 + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{4\pi}\right) = \operatorname{div}(\mathbf{q}_1 + \mathbf{S}_1),$$

$$P_1 = \frac{\varkappa}{M}(1+z)(\rho_0 T_1 + \rho_1 T_0),$$

$$-i\omega\mathbf{B}_1 = \operatorname{rot}(\mathbf{v}\times\mathbf{B}_0) -$$

$$-\frac{c^2}{4\pi}\left\{\operatorname{rot}\left(\frac{1}{\sigma_0}\operatorname{rot}\mathbf{B}_1\right) - \frac{3}{2}\operatorname{rot}\left(\frac{T_1}{\sigma_0 T_0}\operatorname{rot}\mathbf{B}_0\right)\right\},$$

$$\mathbf{q}_1 = \frac{5}{2}\mathbf{v}P_0 + \frac{1}{4\pi}\mathbf{B}_0\times(\mathbf{v}\times\mathbf{B}_0) - \frac{c^2}{(4\pi)^2\sigma_0}\left\{\mathbf{B}_1\times\operatorname{rot}\mathbf{B}_0 + \right.$$

$$\left. + \mathbf{B}_0\times\operatorname{rot}\mathbf{B}_1 - \frac{3}{2}\frac{T_1}{T_0}\mathbf{B}_0\times\operatorname{rot}\mathbf{B}_0\right\},$$

$$\mathbf{S}_1 = -\frac{16}{3}\sigma^0 T_0^3 l_0 \times$$

$$\times\left\{\nabla T_1 + \left(\frac{\partial\ln l_0}{\partial T_0}T_1 + \frac{\partial\ln l_0}{\partial\rho_0}\rho_1\right)\nabla T_0\right\}. \tag{3.1}$$

It is very difficult to perform a general analysis of (3.1), and we use first the geometrical-optics approximation [4] to investigate how the various processes affect the oscillations, i.e., for oscillations of wavelength less than the characteristic length for plasma inhomogeneity:

$$k_x x_p \sim \frac{x_p}{\lambda_x} \gg 1 \tag{3.2}$$

Here $\lambda_x \sim h_x^{-1}$ is the wavelength of the oscillations in the direction of the inhomogeneity. We must also have $\lambda_x \gg l_0$ for the radiative-transfer approximation

*Note that z is only silightly dependent on T, as $z = T^\beta$, with $\beta \le 0.5$, while $z \approx 2$. However, this T dependence of z is neglected below.

to apply. We get the following dispersion equations (eikonal equations [4]) from (3.1) in the zeroth approximation of geometrical optics:

$$1+i\frac{c^2k^2}{4\pi\sigma_0\omega}=0,$$

$$\omega^2-k^2v_a^2\cos^2\alpha+i\frac{c^2k^2\omega}{4\pi\sigma_0}=0,\qquad v_a^2=\frac{B_0^2}{4\pi\rho_0},$$

$$\frac{32}{9}\frac{ik^2}{\omega}\sigma^0T_0^3l_0\left\{1-\frac{k^2(v_a^2+v_s^2)}{\omega^2}+\frac{k^4v_a^2v_s^2\cos^2\alpha}{\omega^4}+\right.$$

$$\left.+\frac{ic^2k^2}{4\pi\sigma_0\omega}\left(1-\frac{k^2v_s^2}{\omega^2}\right)\right\}+\rho_0\frac{\varkappa(1+z)}{M}\left\{1-\frac{k^2}{\omega^2}\left(v_a^2+\frac{5}{3}v_s^2\right)+\right.$$

$$\left.+\frac{5}{3}\frac{k^4v_a^2v_s^2\cos^2\alpha}{\omega^4}+\frac{ic^2k^2}{4\pi\sigma_0\omega}\left(1-\frac{5}{3}\frac{k^2v_s^2}{\omega^2}\right)\right\}=0,$$

$$v_s^2=\frac{\varkappa(1+z)T_0}{M}. \qquad (3.3)$$

Here α is the angle between the magnetic field and the direction of wave propagation, v_s is the speed of isothermal sound in the plasma, and v_a is the Alfvén velocity. Note that (3.3) essentially describes the oscillation spectrum of a homogeneous magnetically active plasma with allowance for radiative transfer and finite conductivity. The first equation describes the penetration of a transverse (vortex) field into the plasma, while the second is the dispersion equation for the Alfvén waves, and the third equation corresponds to fast and slow magnetosonic waves. It is readily shown that the oscillations described by these equations are damped with time ($\gamma = \operatorname{Im}\omega < 0$), and for weak radiative transfer ($k^2 f_0 \sigma^\circ T_0^4 \ll P_0\omega$) they are of adiabatic type, with the damping factor $\gamma \sim l_0$, whereas in the other limit of strong radiative conduction they are isothermal, and the damping factor is $\gamma \sim l_0^{-1}$.

The discharge is therefore stable in the zeroth approximation of geometrical optics, which means that the discharge is stable against perturbations whose wavelength is substantially less than the scale of the plasma inhomogeneity; however, instability may occur for wavelengths $\lambda_x \gtrsim x_p$, and the geometrical-optics approximation is not applicable to these. The frequencies (and hence the growth factors) for these must satisfy $\omega \lesssim (v_s + v_a)/x_p \approx v_s/x_p$ (since $v_a \approx v_s$ for equilibrium). A basic requirement is efficient radiation from the discharge, which, if $x_T^2 \gg x_p^2$, occurs when radiative transfer is rapid. This inequality allows us to neglect inhomogeneity in T_0 relative to the inhomogeneity in density, pressure, and magnetic field in (3.1). If, in addition, we have

$$\frac{\sigma^\circ T_0^4 l_0}{\omega p_0 x_p^2}\approx\frac{x_T^2}{x_p^2}\frac{c^2}{\sigma_0 v_s x_p}\gg 1, \qquad (3.4)$$

the temperature will relax during the oscillations because the radiative transfer is rapid, so we can put $T_1 = 0$ in (3.1), and the system reduces to a single fourth-order differential equation for

$$v=p_1+\frac{\mathbf{B}_0\cdot\mathbf{B}_1}{4\pi},$$

though this is still quite complex. The equation can be analyzed in two opposed limiting cases:

a) $c^2\Delta B_1 \gg 4\pi\sigma_0\omega B_1$ (i.e., $\sigma_0 \to 0$), when it becomes

$$(\omega^2+v_s^2\Delta)v=0; \qquad (3.5)$$

b) $c^2\Delta \mathbf{B}_1 \ll 4\pi\sigma\omega\mathbf{B}_1$ (i.e., $\sigma_0 \to \infty$), when it becomes

$$[\omega^4+(v_a^2+v_s^2)\omega^2\Delta-k_y^2v_s^2v_a^2\Delta]\,v+$$

$$+(\omega^2-k_y^2v_a^2)\times\left[\frac{B_0}{4\pi}\omega^2\frac{\partial}{\partial x}\left(\frac{B_0}{\rho_0}\frac{1}{\omega^2-k_y^2v_a^2}\right)+\right.$$

$$\left.+\frac{2k_y^2v_s^2}{\omega^2-k_y^2v_a^2}\frac{B_0}{4\pi\rho_0}\frac{\partial B_0}{\partial x}\right]\frac{\partial v}{\partial x}=0. \qquad (3.6)$$

To (3.5) and (3.6) we have to add boundary conditions, which can be derived from the conservation of the total current and from the restriction on the acceleration of the plasma boundary in perturbations. If $k_y = 0$ but $k_z \neq 0$ (instabilities of constriction type), these boundary conditions follow directly from (3.1) as

$$v=\frac{\partial v}{\partial x}=0 \qquad \text{for } x=x_p \qquad (3.7)$$

Now (3.5) has only a positive spectrum of eigenvalues ω^2 for any nondissipative boundary conditions, and this corresponds to stable oscillations (sound waves). The situation is different for oscillations described by (3.6) whose fundamental modes can be unstable under certain conditions, as we shall see.

The substitutions

$$v=y\exp\left(-\frac{1}{2}\int^x\varphi(x)\,dx\right),$$

$$\varphi(x)=\frac{B_0}{4\pi}\frac{\omega^2-k_y^2v_a^2}{\omega^2(v_a^2+v_s^2)-k_y^2v_a^2v_s^2}\times$$

$$\times\left[\frac{\partial}{\partial x}\left(\frac{B_0}{\rho_0}\frac{\omega^2}{\omega^2-k_y^2v_a^2}\right)+\frac{2k_y^2v_s^2}{\omega^2-k_y^2v_a^2}\frac{1}{\rho_0}\frac{\partial B_0}{\partial x}\right] \qquad (3.8)$$

convert (3.6) to the form of the Schrödinger equation;

$$\frac{d^2y}{dx^2}+\left\{\frac{\omega^4}{\omega^2(v_s^2+v_a^2)-k_y^2v_s^2v_a^2}-\right.$$

$$\left.-\frac{1}{2}\frac{\partial\varphi}{\partial x}-\frac{\varphi^2}{4}-(k_y^2+k_z^2)\right\}y=0. \qquad (3.9)$$

Eigenvalues of ω^2 having $\operatorname{Im}\omega > 0$ correspond to unstable oscillations. For modes with $k_y = 0$ (constrictions), (3.7) amounts to $y(x = \pm x_p) = 0$, and so we find that such oscillations can be unstable only if

$$U(x)=k_z^2+\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{B_0}{4\pi(v_s^2+v_a^2)}\frac{\partial}{\partial x}\frac{B_0}{\rho_0}\right]+$$

$$+\frac{1}{4}\left[\frac{B_0}{4\pi(v_s^2+v_a^2)}\frac{\partial}{\partial x}\frac{B_0}{\rho_0}\right]^2<0. \qquad (3.10)$$

We always have $U(x) > 0$ for an equilibrium discharge described by (2.2) and (2.3), i.e., such a discharge is stable [5]; only a nonequilibrium discharge can be unstable, when some one of the conditions of (2.1) is not met. For instance, $\mathbf{E}_0$ penetrates only fairly slowly into the plasma if $\sigma_0 \to \infty$ (relative to the compression rate, that is), and a pronounced skin effect can occur for the current $\mathbf{j}_0(x)$, whereas P_0 has time to equilibrate with the magnetic pressure. Then the boundary conditions of (3.7) do not apply, in general; a local analysis of the stability via (3.10) shows that the discharge can be unstable if $3P_0'^2 < 2P_0P_0''$, and the growth factor can be $\gamma \sim v_s/x_p$.

It is more tedious to analyze the stability of the solutions to (3.6) for modes with $k_y \neq 0$, but the main conclusion for modes with $k_y = 0$ applies [5] to them also, namely that a planar equilibrium discharge is stable against such perturbations, whereas a nonequilibrium discharge may be unstable, the maximum growth rate being $\gamma_{max} \leqslant v_s/x_p$. We therefore see that a real discharge is stable if the time for the instability to grow is greater than the equilibration time (time for $\mathbf{E}_0$ to penetrate):

$$\frac{c^2}{4\pi\sigma_0 v_s} > x_p. \qquad (3.11)$$

This inequality is met if $N \leqslant 10^{28} A E_0^2 T_0^{-2}$, in which A is the atomic number of the ions.

4. Stability of a cylindrical discharge. The above analysis is readily extended to a plasma cylinder carrying a current, with the perturbed quantities represented as functions of time and coordinates by $f(r) \times (\exp(-i\omega t + im\varphi + ik_z z)$, in which m is azimuthal wave number. In the geometrical-optics approximation, the spectrum of the oscillations is the same (apart from the trivial substitution $k_y \to m/r$) so we consider only the fundamental modes, to which that approximation is not applicable. We again assume (3.4) to be met (with x_p and x_T replaced by r_p and r_T, respectively) and put $T_1 = 0$; then (3.1) becomes a single fourth-order differential equation for

$$v = \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{2\pi r} + \frac{\partial}{\partial r}\left(p_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{4\pi}\right).$$

If σ_0 is high, that is $c^2 \Delta B_1 \ll 4\pi\sigma_0\omega B_1$ (i.e., $\sigma_0 \to \infty$), the equation for v becomes of second order:

$$\left(\frac{r}{2}\frac{\partial}{\partial r} + \frac{k_z^2 r^2 v_a^2}{\omega^2 r^2 - m^2 v_a^2}\right) \times$$
$$\times \frac{r^2(\omega^2 r^2 - m^2 v_a^2)}{k_z^2 v_s^2 r^4 \omega^2 + (m^2 v_s^2 - \omega^2 r^2)(\omega^2 r^2 - m^2 v_a^2 - k_z^2 r^2 v_a^2)} \times$$
$$\times \left[\frac{\omega^2 B_0}{4\pi}\frac{\partial}{\partial r}\frac{r^2 B_0}{\rho_0(\omega^2 r^2 - m^2 v_a^2)} - \right.$$
$$- \frac{m^2 v_s^2}{r}\frac{B_0}{4\pi}\frac{\partial}{\partial r}\frac{r}{\rho_0(\omega^2 r^2 - m^2 v_a^2)} +$$
$$\left. + \frac{v_s^2}{r}\frac{\partial}{\partial r}\frac{\omega^2 r^3}{\omega^2 r^2 - m^2 v_a^2}\right] v -$$
$$- \left(\frac{B_0}{4\pi}\frac{\partial}{\partial r}\frac{r^2 B_0}{\rho_0(\omega^2 r^2 - m^2 v_a^2)} + \frac{r}{2}\right) v = 0, \qquad (4.1)$$

The boundedness of the perturbations for $r \leq r_p$ is sufficient to define uniquely the spectrum of eigenvalues of ω^2. This requirement is equivalent to the conservation of total current and the bounded acceleration used above. Equation (4.1) has been examined in detail [6] for modes with m = 0 and $k_z \neq 0$ (constrictions), where it was shown that an equilibrium cylindrical discharge has instabilities whose growth rates in the long-wave limit ($k_z r_p < 1$) are

$$\gamma^2 = -\omega^2 = \frac{4k_z^2 v_s^2}{(\pi n + 0.75\pi)^2} \leqslant \frac{v_s^2}{r_p^2},$$
$$\gamma^2 = -\omega^2 = 2\sqrt{3}\frac{|k_z| v_s^2}{r_p} \leqslant \frac{v_s^2}{r_p^2}, \qquad (4.2)$$

respectively, for the higher ($n \geq 1$) and fundamental ($n = 0$) modes. Analogous formulas describe the instability at shorter wavelengths ($k_z r_p > 1$), for which $\gamma \leqslant v_s/r_p$. It can be shown that the growth rates are of the same order for modes with $m \neq 0$ (kinks) in an ideally conducting plasma.

Consider now a poorly conducting plasma containing a current, when $c^2 \Delta B_1 \gg 4\pi\sigma_0\omega B_1$ (i.e., $\sigma_0 \to 0$); we show that only the fundamental mode (n = 0) can be unstable here. We consider only constrictions with m = 0. Here it is convenient to start from (3.1) in the form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - k_z^2\right)\chi = 0,$$
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\omega^2}{v_s^2} - k_z^2\right)u = -\left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{r\omega^2}{2v_s^2}\right)\chi,$$
$$\chi = \frac{B_0 B_{1\varphi}}{2\pi_r}, \quad u = p_1 + \frac{B_0 B_{1\varphi}}{4\pi}. \qquad (4.3)$$

Then the boundary conditions take the following form, being equivalent to current conservation and restricted boundary acceleration:

$$u = \left(\frac{\partial u}{\partial r} + \chi\right) = 0 \quad \text{for } r = r_p \qquad (4.4)$$

System (4.3), (4.4) reduced to the following dispersion equation in the region $\omega^2 < k_z^2 v_s^2$, which contains also unstable solutions ($\omega^2 < 0$):

$$I_0(\beta r_p)\left[I_1(\alpha r_p) + \frac{d}{dr_p}\Phi(r_p)\right] - \beta I_1(\beta r_p)\Phi(r_p) = 0,$$
$$\Phi(r_p) = -\frac{2\alpha v_s^2}{\omega^2} I_0(\alpha r_p) + \frac{r_p}{2} I_1(\alpha r_p),$$
$$\alpha = |k_z|, \quad \beta = \left|\sqrt{k_z^2 - \omega^2/v_s^2}\right|. \qquad (4.5)$$

In the long-wave limit, where $\alpha r_p \ll 1$ and $\beta r_p \ll$ $\ll 1$, (4.5) permits unstable oscillations only for the fundamental (n = 0), the growth rate being as for an ideally conducting plasma, i.e., being defined by the second expression in (4.2). Short-wave oscillations are also unstable, but they present no great hazard, as they are damped out within the volume of the plasma.

5. Discussion and conclusions. The results are discussed as regards use of such a discharge as a light source for laser pumping. Primary requirements on a light source are a high temperature in the emitting surface and a reasonably prolonged stable period. This is why a discharge of the above type was chosen. Formulas (2.2) and (2.3) define the equilibrium state, with $r_T \gg r_p \gg l$. These inequalities are put in the form of (2.5) for highly ionized atoms and T_0 of 3-10 eV, and they are met by densities $10^{17} < N < 10^{20}$ cm^{-3} (for E_0 of 0.3–1 electrostatic cgs units).

The stability anaysis shows the high radiative conductivity, so temperature perturbations relax rapidly if (3.4) is met, and instabilities of hot-spot type are absent. Long- and short-wave oscillations are unstable in a plasma of high conductivity, whereas one of low conductivity has only long-wave instabilities ($k_z r_p \ll 1$) when (3.11) is obeyed. A planar discharge is then completely stable, while a cylindrical discharge is unstable only for the fundamental mode.

This means that it is best to use as a light source a discharge at the surface of a hollow cylinder of radius R of 5-10 cm, where the maximum growth rate is for instabilities. In the relevant temperature range, a heavy gas ($v_s \approx 10^5$ cm/sec) has then $\gamma_{max} \approx (2-1) \cdot 10^4$ sec^{-1}, which corresponds to a stable period of 50-100 μsec.

Formulas (2.2)–(2.4) become meaningless for $x = x_p$ (or $r \approx r_p$), because the Rosseland length increases rapidly as the plasma density decreases, and it is no longer correct to use the above radiative-conduction approximation. The discharge in a dense opaque plasma is always surrounded by a layer of transparent plasma carrying a current. If $x_p \gg l$, this layer makes no great contribution to the energy balance or to the character of the radiation, but it can play a considerable part in the stability.

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REFERENCES

1. B. B. Kadomtsev, "Hydromagnetic stability of a plasma," Voprosy teorii plasmy, 2, 132, 1963.
2. L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media [in Russian], Gostekhizdat, Moscow, 1957.
3. Ya. B. Zel'dovich and Yu. P. Raizer, The Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena [in Russian], Fizmatgiz, Moscow, 1963.
4. A. A. Rukhadze and V. P. Silin, "The method of geometrical optics in the electrodynamics of an inhomogeneous plasma", Usp. fiz. nauk, 82, 499, 1964.
5. M. Kruskal and M. Schwarzschild, "Some instabilities of a completely ionized plasma," Proc. Roy. Soc., A223, p. 348, 1952.
6. B. A. Trubnikov, "Instability of a plasma cylinder," collection: Plasma Physics and Controlled Thermonuclear Reactions, Vol. 1 [in Russian], 289, 1958.

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